Comment on the Ghost State in the Lee Model

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Abstract

We examine the ghost state in the Lee model, and give the physically meaningful interpretatation for norm of the ghost state. According to this interpretation, the semi-positivity of the norm is guaranteed.

§1. Introduction

The field theory is an essential tool to study particle physics, because the particle theory only can treat the creation and the annihilation of particles. While infinite quantities appear in the calculation of the field theory, and the renormalization of physical quantities becomes necessary. But this renormalization scheme is an approximate and perturvative one. Then the Lee model[1, 2] is introduced to study the essense of the renormalization scheme in an exact form, though this is a toy model in a sense that it restricts the interaction in a quite special form and also it is not relativistic invariant.

In this Lee model, there is difficulty that there appears the negative norm state, the ghost state, in the strong coupling region, which contradict with the physically meaningful condition.

In this paper, we first review the Lee model and next give the physically meanigful interpretation for the norm of the ghost state. According to this interpretation, the semi-positivity of the norm is guaranteed.

§2. Review of the Lee Model

We breafly review the Lee model [1, 2]. The Hamiltonian of the Lee model is given by

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$$H_0 = m_{V_0} \int d\vec{p} V^\dagger(\vec{p}) V(\vec{p}) + m_N \int d\vec{q} N^\dagger(\vec{q}) V(\vec{q}) + \int d\vec{k} \omega_k \theta^\dagger(\vec{k}) \theta(\vec{k}), \tag{1}$$

$$H_I = \frac{g_0}{(2\pi)^{3/2}} \int d\vec{k} d\vec{p} \frac{f(\omega_k)}{\sqrt{2\omega_k}} \left\{ V^\dagger(\vec{p}) N(\vec{p} - \vec{k}) \theta(\vec{k}) + N^\dagger(\vec{p} - \vec{k}) V(\vec{p}) \theta^\dagger(\vec{k}) \right\}, \tag{2}$$

where $\omega_k = \sqrt{\vec{k}^2 + \mu^2}$, and m_{V_0}, m_N, μ represents bare masse of V, N, θ particles respectively. The vacuum state $|0>$ is defined by

$$V(\vec{p})|0> = N(\vec{q})|0> = \theta(\vec{k})|0> = 0. \tag{3}$$

One N partcle state $N^\dagger(\vec{q})|0>$ and one θ particle state $\theta^\dagger(\vec{k})|0>$ respectively is the trivial eigenstate of the total Hamiltonian

$$(H_0 + H_I) N^\dagger(\vec{q})|0> = m_N N^\dagger(\vec{q})|0>,$$
$$(H_0 + H_I) \theta^\dagger(\vec{k})|0> = \omega_k \theta^\dagger(\vec{k})|0> . \tag{4}$$

While one V particle state $V^\dagger(\vec{p})|0>$ itself is not the eigenstate of the total Hamiltonian, but the linear combination of $V^\dagger(\vec{p})|0>$ and $\theta^\dagger(\vec{k}) N^\dagger(\vec{p} - \vec{k})|0>$ gives the eigenstate. We first examine what happens after applying the Hamiltonian to the one V particle state $V^\dagger(\vec{p})|0>$

$$H_0 V^\dagger(\vec{p})|0> = m_{V_0} V^\dagger(\vec{p})|0>,$$
$$H_I V^\dagger(\vec{p})|0> = \frac{g_0}{(2\pi)^{3/2}} \int d\vec{k} \frac{f(\omega_k)}{\sqrt{2\omega_k}} \theta^\dagger(\vec{k}) N^\dagger(\vec{p} - \vec{k})|0> . \tag{5}$$

We next examine what happens after applying the Hamiltonian to the state $\theta^\dagger(\vec{k}) N^\dagger(\vec{p} - \vec{k})|0>$

$$H_0 \theta^\dagger(\vec{k}) N^\dagger(\vec{p} - \vec{k})|0> = (\omega_k + m_N) \theta^\dagger(\vec{k}) N^\dagger(\vec{p} - \vec{k})|0>,$$
$$H_I \theta^\dagger(\vec{k}) N^\dagger(\vec{p} - \vec{k})|0> = \frac{g_0}{(2\pi)^{3/2}} \frac{f(\omega_k)}{\sqrt{2\omega_k}} V^\dagger(\vec{p})|0> . \tag{6}$$

Then we try to find the renormalized V particle state $|V^{\text{ren}}(\vec{p})>$, which is the eigenstate of the total Hamiltonian in the form

$$|V^{\mathrm{ren}}(\vec{p}) >= Z_V^{1/2} \left\{ V^\dagger(\vec{p})|0 > + \int d\vec{k}\Phi(\vec{k})\theta^\dagger(\vec{k})N^\dagger(\vec{p}-\vec{k})|0 > \right\}, \tag{7}$$

and we denote the eigenvalue of the total Hamiltonian to be m_V in the form

$$(H_0 + H_I)|V^{\mathrm{ren}}(\vec{p}) >= m_V|V^{\mathrm{ren}}(\vec{p}) > . \tag{8}$$

Straightforward calculation gives

$$(H_0 + H_I)|V^{\mathrm{ren}}(\vec{p}) >= Z_V^{1/2} \left\{ m_{V_0}V^\dagger(\vec{p})|0 > + \frac{g_0}{(2\pi)^{3/2}} \int d\vec{k}\frac{f(\omega_k)}{\sqrt{2\omega_k}}\theta^\dagger(\vec{k})N^\dagger(\vec{p}-\vec{k})|0 > \right.$$
$$+ \int d\vec{k}\Phi(\vec{k})\left((\omega_k + m_N)\theta^\dagger(\vec{k})N^\dagger(\vec{p}-\vec{k})|0 > + \frac{g_0}{(2\pi)^{3/2}}\frac{f(\omega_k)}{\sqrt{2\omega_k}}V^\dagger(\vec{p})|0 > \right) \right\}$$
$$= Z_V^{1/2} \left\{ \left(m_{V_0} + \frac{g_0}{(2\pi)^{3/2}} \int d\vec{k}\Phi(\vec{k})\frac{f(\omega_k)}{\sqrt{2\omega_k}} \right) V^\dagger(\vec{p})|0 > \right.$$
$$\left. + \int d\vec{k}\left(\frac{g_0}{(2\pi)^{3/2}}\frac{f(\omega_k)}{\sqrt{2\omega_k}} + (\omega_k + m_N)\Phi(\vec{k}) \right)\theta^\dagger(\vec{k})N^\dagger(\vec{p}-\vec{k})|0 > \right\}. \tag{9}$$

From Eq.(8), we obtain the relation

$$m_V = m_{V_0} + \frac{g_0}{(2\pi)^{3/2}} \int d\vec{k}\Phi(\vec{k})\frac{f(\omega_k)}{\sqrt{2\omega_k}}, \tag{10}$$

$$m_V\Phi(\vec{k}) = \frac{g_0}{(2\pi)^{3/2}}\frac{f(\omega_k)}{\sqrt{2\omega_k}} + (\omega_k + m_N)\Phi(\vec{k}). \tag{11}$$

From Eq.(11), we have

$$\Phi(\vec{k}) = \frac{g_0}{(2\pi)^{3/2}}\frac{f(\omega_k)}{\sqrt{2\omega_k}}\frac{1}{m_V - m_N - \omega_k}. \tag{12}$$

Substitution this $\Phi(\vec{k})$ into Eq.(10), we have

$$m_V = m_{V_0} + \frac{g_0^2}{(2\pi)^3} \int d\vec{k}\frac{f^2(\omega_k)}{2\omega_k}\frac{1}{m_V - m_N - \omega_k}, \tag{13}$$

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which gives the mass renormalization relation

$$\delta m_V = m_V - m_{V_0} = \frac{g_0{}^2}{(2\pi)^3} \int d\vec{k} \frac{f^2(\omega_k)}{2\omega_k} \frac{1}{m_V - m_N - \omega_k}. \tag{14}$$

The norm of the renormalized vector state $|V^{\mathrm{ren}}(\vec{p}) >$ gives the condition

$$1 = Z_V \left\{ 1 + \int d\vec{k} |\Phi(\vec{k})|^2 \right\}. \tag{15}$$

Substituting the explicit form of $\Phi(\vec{k})$, we have

$$Z_V^{-1} = 1 + \frac{g_0{}^2}{(2\pi)^3} \int d\vec{k} \frac{f^2(\omega_k)}{2\omega_k} \frac{1}{(m_V - m_N - \omega_k)^2}. \tag{16}$$

We connect the bare coupling g_0 and the renormalized coupling g in the form $g^2 = Z_V g_0{}^2$, which comes from the condition that the $N + \theta \to N + \theta$ scattering amplitude is expressed only with renormalized quantities.

Using this relation, we have

$$Z_V^{-1} = 1 + Z_V^{-1} \frac{g^2}{(2\pi)^3} \int d\vec{k} \frac{f^2(\omega_k)}{2\omega_k} \frac{1}{(m_V - m_N - \omega_k)^2}, \tag{17}$$

which can be rewritten in the form

$$Z_V^{-1} \left(1 - \frac{g^2}{(2\pi)^3} \int d\vec{k} \frac{f^2(\omega_k)}{2\omega_k} \frac{1}{(m_V - m_N - \omega_k)^2} \right) = 1. \tag{18}$$

Thus we can solve Z_V^{-1} in the form

$$Z_V^{-1} = 1 / \left(1 - \frac{g^2}{(2\pi)^3} \int d\vec{k} \frac{f^2(\omega_k)}{2\omega_k} \frac{1}{(m_V - m_N - \omega_k)^2} \right), \tag{19}$$

which gives

$$Z_V = 1 - \frac{g^2}{(2\pi)^3} \int d\vec{k} \frac{f^2(\omega_k)}{2\omega_k} \frac{1}{(m_V - m_N - \omega_k)^2}. \tag{20}$$

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§3. Physical Interpretation for the Norm of the Ghost State

Because Z_V is the probability to exist bare V-particle state $|V(\vec{p}) >$ in the renormalized V-particle state $|V^{\text{ren}}(\vec{p}) >$,

$$0 \leq Z_V \leq 1, \tag{21}$$

must be satisfied.

In the strong coupling region,

$$\frac{g^2}{(2\pi)^3} \int d\vec{k} \frac{f^2(\omega_k)}{2\omega_k} \frac{1}{(m_V - m_N - \omega_k)^2} > 1, \tag{22}$$

Z_V becomes negative by using the relation

$$Z_V = 1 - \frac{g^2}{(2\pi)^3} \int d\vec{k} \frac{f^2(\omega_k)}{2\omega_k} \frac{1}{(m_V - m_N - \omega_k)^2}, \tag{23}$$

under the standard interpretation, but which contradict with the physically meaningful condition Eq.(21), which is the origin of the ghost problem in the Lee model.

We can give another physically meaningful interpretation for that norm.

Let's denote

$$x = \frac{g^2}{(2\pi)^3} \int d\vec{k} \frac{f^2(\omega_k)}{2\omega_k} \frac{1}{(m_V - m_N - \omega_k)^2}, \tag{24}$$

and we interprete Eq.(19) as

$$Z_V^{-1} = 1/(1 - x) = 1 + x + x^2 + x^3 + x^4 + \cdots \quad (x > 1). \tag{25}$$

Then we have $Z_V^{-1} = 1 + x + x^2 + x^3 + x^4 + \cdots = +\infty$ (with $x > 1$), which means $Z_V = 1/(1 + x + x^2 + x^3 + x^4 + \cdots) = 0$ (with $x > 1$) instead of taking negative value.

We interprete that the expression $1/(1-x)$ (with $x > 1$) is the symbol to express infinity here, that is, we interprete $1/(1-x) = 1 + x + x^2 + x^3 + x^4 + \cdots$ (with $x > 1$). We can make an intuitive interpretation why $1/(1 - x)$ (with $x > 1$) can express infinity. If we devide 1 by $0 = (1 - 1)$, we have infinity $1/(1 - 1) = 1 + 1 + 1 + \cdots$. Then if we devide 1 by $(1 - x)$ (with $x > 1$) "the number with the magnitude being smaller than 0", we obtain another

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infinity $1/(1-x) = 1 + x + x^2 + x^3 + x^4 + \cdots$ (with $x > 1$), which is the larger infinity than the former infinity $1/(1-1) = 1 + 1 + 1 + \cdots$. Of course, there is no actual number with "the magnitude being smaller than 0". It is the number living in the world of ideal, that is, it appears only in connection with infinity.

§5. Summary

We examine the ghost state in the Lee model, and give the physically meanigful interpretatation for norm of the ghost state. According to this interpretation, the semi-positivity of the norm is guaranteed. Depending on the situation, $1/(1-x)$ (with $x > 1$) represent finite number or infinity of the form $1/(1-x) = 1 + x + x^2 + x^3 + x^4 + \cdots$. For the problem of the ghost state in the Lee model, we interprete $1/(1-x)$ (with $x > 1$) as the symbol to express infinity, and obtain the physically meaningful result. It may be useful to classify many infinities by symbolically express these infinities in the form $1/(1-x)$ (with $x > 1$).

References

[1] T.D.Lee, *Phys. Rev.* **95**(1954), 1329.

[2] W. Heisenberg,:*Nucl. Phys.* **4**(1957), 532.